Generating Functions

1. Introduction

Generating Functions allow us to use powerful algebra techinques to approach difficult counting problems. We can also refer to a generating function as a **power series** or an "infinte polynomial". It makes combinatorics almost like algebra.

Let's say we have the 4th row of Pascal's Triangle. The generating function for that row would be $x^4 + 4x^3 + 6x^2 + 4x + 1$.

2. Examples of Generating Functions

Example: From AOPS Inter C and P

In how many ways can we roll three dice to get the sum of 9?

You could approach this problem with some ugly casework or you could use the more elegant solution of turning it into an algebra problem. The generating function for a die would be $x^1 + x^2 + x^3 + x^4 + x^5 + x^6$ and then for three die it would ne this, $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^3$. We would just expand this all out since we don't have a good algebriac tool we just have to expand it out. $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 25x^6 + 27x^7 + 27x^8 + 25x^9 + ...$ In the expression, the coefficent of x^9 is 25 so that is our answer.

We will learn some methods on how to make simplification easier.

We can think of the Binomial Theorem as a generating function.

We know that the coefficent of x^k is $\binom{n}{x}$. It counts the way to pick x items from n items. So we can view $(x + 1)^n$ as the ways to choose subsets from a set of n numbers. As we saw with our previous example, the coefficent of x^k counts something that depends on k.

Usually, the generating function arising from the Binomial Theorem doesn't help much in solving problems since we can often solve such problems by other means.

Although, at times it can help clarify a solution.

Example: From AOPS Inter C and P

Joe is ordering 3 hot dogs at Alpine City Beef. He can choose among five toppings for each hot dog: ketchup, mustard, relish, onions, and sauerkraut. In how many different ways can he choose 6 toppings total for his three hot dogs?

Let's look at each hotdog individually. He has $\binom{5}{0}$ ways to choose 0 toppings, $\binom{5}{1}$ ways to choose 1 topping and so forth.

This means that we have $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$. That is $(x + 1)^5$. For 3 hotdogs we can make this, $(x + 1)^{15}$. We want the coefficent of x^6, which would be $\binom{15}{6}$ by the binomial theorem.
That was pretty nice!

3. Using Infinite Power Series

Example

There are 30 gifts to be distrubuted among a group of 50 children on a field trip. Each child may get more than one gift. How many ways are there to distribute the 30 gifts among the 50 children?

You could use stars and bars here, but for the sake of learning we are going to use generating functions. Each student can get 0, 1, 2 and so on gifts. While each student can get a maximum of 30 gifts so we should use, $(1 + x + ...x^{30})$ as our generating function for each student, but this will actually make our calculations harder.

So instead, we turn $(1 + x + ...x^{30})$ into $(1 + x + x^2...)$ or into a infite polynomial with a ratio of x. This will give us, $(1 + x + x^2...^{50})$ or $\frac{1}{(1-x)^{50}}$. This can be written into a form where we can use the binomial theorem like in the examples above.

That expression is equivalent to $(1 - x)^{-50}$. We want to find the coefficent of x^{30}. This means that we have $\binom{-50}{30}$ or $\binom{79}{30}$.

Using infinite series instead of a constricted series can serve to be very useful and needed in many cases.

Something that we should observe as in the previous problem is the form $(1 + x + ...x^n)$. We had n = 30 in the previous exmaple but let's study the general form. Observe that...

$$(1 + x + x^2 + x^3...)^1 = 1 + x + x^2 + x^3$$

$$(1 + x + x^2 + x^3...)^2 = 1 + 2x + 3x^2 + 4x^3...$$

$$(1 + x + x^2 + x^3...)^3 = 1 + 3x + 6x^2 + 10x^3...$$

Let's rewrite this with binomial coefficents.

$$(1 + x + x^2 + x^3...)^1 = \binom{0}{0} + \binom{1}{0}x + \binom{2}{0}x^2 + \binom{3}{0}x^3...$$

$$(1 + x + x^2 + x^3...)^2 = \binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \binom{4}{1}x^3...$$

$$(1 + x + x^2 + x^3...)^3 = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \binom{5}{2}x^3...$$

There is a clear pattern, in fact we can generalize our observations to the equation below...

$$(1 + x + x^2 + x^3...)^n = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \binom{n+2}{n-1}x^3...$$

This is a useful fact, because infinite power series can come up a lot and this is a quick way to figure out a binomial coefficent for a certain x^k.

Let's apply this knowledge in a new problem.

Example

Find the number of positive integer solutions to a + b + c + d = 10.

There are many ways to approach this problem. They may include using distributions or using bashy methods, but generating functions works as well!

We have to pay attention to the fact that we want positive integer solutions. This just means that we no longer have a x^0 term in our generating function.

So the function for this is $(x + x^2 + x^3 + ...)$ and for each of the 4 integers we have $(x + x^2 + x^3 + ...)^4$. Let's factor out an x to make it look familiar. $x^4(1 + x + x^2 + ..)^4$. We notice that we can just find the coefficent of the x^6 term since we are multiplying on x^4.

From our equations above, we notice that the coefficent we want is $\binom{9}{3}$ which is our answer.

4. Utilizing the Roots of Unity

That seems quite odd. How can one possibly use roots of unity and generating functions together? Well, it's actually not that impossible.

Specifically it's called the Roots of Unity filter. The filer gives us the coefficent of x^n such that n is divisble by k.

$$\frac{P(1) + P(w) + P(w^2)...P(w^{k-1})}{k}$$

where P(x) is a polynomial and 1 through w^k are roots of unity.

The fact of $1 + w + w^2 + ... + w^{k-1} = 0$ will be very helpful in computations. Let's look at an example.

Example: CNCM

How many ways are there to roll 4 standard 6 sided dice such that the sum of the numbers rolled is divisible by 3?

This seems perfect for the roots of unity filter. Our generating function is $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$ and we want to use the filter for k = 3. The generating function will serve as our P(x).

So we write $\frac{P(1)+P(x)+P(x^2)}{3}$ and we just need to compute this. We have that $P(1) = 6^4$, and..

$$P(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4 = (x + x^2 + 1 + x + x^2 + 1)^4 = 0.$$
$$P(x^2) = (x^2 + x^4 + x^6 + x^8 + x^10 + x^12)^4 = (x + x^2 + 1 + x + x^2 + 1)^4 = 0.$$

So our answer is $6^4/3$ which is equal to 432 ways in total.

Pretty unique calculation if you ask me!

It's a simple idea, but very powerful.

This handout was meant to be a brief overview of what you can do with generating functions, so I hope you enjoyed and test out your new skills with some problems down below.

Practice Problems

1. (2007 HMMT COMBO) Let S be the set of all triples (a,b,c) of positive integers such that a + b + c = 15. Compute

$$\Sigma_{i=1}^{100} i$$

2. (AMC 12A 2015) For each positive integer n, let $S(n)$ be the number of sequences of length n consisting solely of the letters A and B, with no more than three As in a row and no more than three Bs in a row. What is the remainder when $S(2015)$ is divided by 12?

3. (AMC 2006 12A) The expression

$$(x+y+x)^{2006} + (x-y-z)^{2006}$$

is simplified and expanded by combining like terms. How many terms are there in total?

4. (AOPS Book) Compute

$$\Sigma_{i=0}^{\infty} k(\frac{1}{3})^k$$

using generating functions. (Hint: Which generating function looks like $\Sigma_{i=0}^{\infty} k(x)^k$?)