Problems and Solutions in Advanced Graph Theory

Niyathi Kukkapalli[1]

[1]University of Delaware, Delaware, USA

November 10, 2023

Abstract

A compilation of selected problems and solutions from my time working with Professor Sebastian Cioba at University of Delaware's Math Department in Summer of 2023. In the process of editing the first version of, *A First Course in Graph Theory and Combinatorics*, our team (1 graduate student and 4 undergraduates) worked through most of the book in preparation for our research.

1 Problems and Solutions

4.6.1 **Show that a graph X with n vertices is connected if and only if** $(A + I_n)^{n-1}$ **has no zero entries, where A is the adjacency matrix of X.**

If X is connected, then we know that there is a path between any two vertices x and y. Any path between these two vertices must have length of at most $n-1$. Therefore, there is going to be at least one walk of length at most $n-1$.

Say that for any two vertices there is a walk of length l connecting them. Then by Theorem 4.1.2 the xy entry in A^l must be positive. After expanding the binomial we are summing all A^l for $0 \leq l \leq n-1$ and since all the entries are non-negative we will end up with a matrix with all non-zero entries.

To prove the right to left direction assume that $(A + I_n)^{n-1}$ has no zero entries. After expanding the binomial we can see the resulting matrix is going to be the sum of all powers of A^l for $0 \leq l \leq n-1$. Since the resulting matrix has non-zero entries there must be at least one matrix A^l for $0 \leq l \leq n-1$ where the xy entry is positive. This means by theorem 4.1.2 there is a walk between any two vertices x and y and therefore the graph is connected.

4.6.2 **Let X be a** k**-regular graph. Show that** λ **is an eigenvalue of its adjacency matrix if and only if** $k - \lambda$ **is an eigenvalue of its Laplacian matrix.**

Let λ be the eigenvalue of the adjacency matrix of the k-regular graph. Then the eigenvalue of L which is $D - A$ will be $k - \lambda$ since k is the eigenvalue for any non zero eigenvector and λ is the eigenvalue of A. The proof of the right to left direction assume that $k - \lambda$ is the eigenvalue of $D - A$ then by the eigenvalue eigenvector equation we have

$$(D - A)v = (k - \lambda)v$$

$$Dv - Av = kv - \lambda v$$

Since $Dv = kv$ for any non-zero v, then λ must be the eigenvalue of A.

4.6.3 **Let X be a k-regular graph. Show that k is the largest eigenvalue of the adjacency matrix A of X. Prove that X is connected if and only if the multiplicity of k is one.**

(1) Because X is a k-regular graph every row of the adjacency matrix will sum up to k. This means that when multiplying the matrix with the all ones vector $\vec{1}$ we get that

$$A\vec{1} = k\vec{1}.$$

Therefore, k is the eigenvalue of A.

Now let λ be an arbitrary eigenvalue of A and let a corresponding eigenvector be u. Because u is an eigenvector, u must have some non-zero entries. By replacing u by $-u$, we may assume that u has at least one positive entry.

Let $j \in \{1, \ldots, n\}$ be such that $u_j = \max\{u_\ell : 1 \leq \ell \leq n\}$. By our previous argument, $u_j > 0$. Therefore,

$$\lambda u_j = (Au)_j = \sum_{\ell=1}^{n} A_{j\ell} u_j = \sum_{\ell:\ell\sim j} u_\ell \leq \sum_{\ell:\ell\sim j} u_j = ku_j.$$

Because $u_j > 0$, we get that $\lambda \leq k$.

(2) Assume that X is connected. Let v be an eigenvector corresponding to the eigenvalue k. Denote by t the largest natural number between 1 and n such that $v_t = \max\{v_\ell : 1 \leq \ell \leq n\}$. From the eigenvalue-eigenvector equation $Av = kv$, we deduce that

$$\begin{aligned} kv_t = (Av)_t &= \sum_{\ell:\ell\sim t} v_\ell \\ &\leq \sum_{\ell:\ell\sim t} v_t \\ &= kv_t. \end{aligned}$$

We deduce that we must have equality in the inequality

$$\sum_{\ell:\ell\sim t} v_\ell \leq \sum_{\ell:\ell\sim t} v_t,$$

and so

$$\sum_{\ell:\ell\sim t} v_\ell = \sum_{\ell:\ell\sim t} v_t,$$

and therefore, $v_\ell = v_t$ for each $\ell \sim t$.

We now prove that $v_s = v_t$ for any $s \in V$. To do this formally, we use induction on $d(s,t)$ the distance between s and t. If this distance is 0, then $s = t$ and clearly, $v_s = v_t$. If $d(s,t) = 1$, then $v_s = v_t$ from our previous work. Let $k \geq 2$. Assume that for any vertex s' with $d(s',t) = k-1$, $v_{s'} = v_t$. We prove that for any vertex s with $d(s,t) = k$, we must have that $v_s = v_t$. Because $d(s,t) = k$, there must be an (s,t)-path of length k. Consider the vertex s' on this path that is adjacent to t. From this choice of s', we get that $d(s',t) \leq k-1$. The triangle inequality implies that $d(s',t) \geq d(s,t) - d(s,s') = k-1$. Thus, $d(s',t) = k-1$. By our induction hypothesis, $v_{s'} = v_t$. Using the first part of our proof, we deduce that $v_s = v_{s'} = v_t$ because s and s' are adjacent and $v_{s'} = v_t$. This finishes our proof.

Hence, all the entries of v are equal and $v = v_1\vec{1}$. This means that the eigenspace of k is one-dimensional and finishes our proof.

An alternative proof can be given using the Laplacian matrix. If X is k-regular, then $L(X) = D(X) - A(X) = kI - A$. Therefore, if λ is an eigenvalue of A, then $k - \lambda$ is an eigenvalue of L with the same multiplicity. Since 0 is an eigenvalue of L with multiplicity one, this means that k must be an eigenvalue of A with multiplicity one. Also, because every eigenvalue of L is non-negative, it follows that every eigenvalue of A is less than or equal to k.

4.6.4 **Let X be a k-regular graph. If λ is an eigenvalue of the adjacency matrix of X, show that $\lambda \geq -k$. If X is connected and k-regular, then prove that X is bipartite if and only if $-k$ is an eigenvalue of its adjacency matrix.**

(1) Let λ be an eigenvalue of the adjacency matrix of X and let u be the corresponding eigenvector. Let $u^* = \max_{i=1}^{n}(|u_i|)$, then for any $0 \leq m \leq n$ such that $|u_m| = u^*$ and if u_m is negative we can take $-u$ as the eigenvector to make u_m positive, we have the following inequality

$$\lambda u_m = (Au)_m = \sum_{j=1}^{n} A_{ij}u_j = \sum_{j\sim m} u_j \geq \sum_{j\sim i} -u_m = -ku_m.$$

(2) Assume X is a connected k-regular bipartite graph then by Exercise 4.6.3 k is an eigenvalue of X. By Theorem 4.3.1 $-k$ is also an eigenvalue of X.
To prove the right to left direction assume that X is a connected k-regular graph and $-k$ is the eigenvalue of the adjacency matrix. We first prove the identity for any vector u, $u^T(kI_n + A)u = \sum_{xy\in E}(u_x + u_y)^2$. The left side of the equation

simplifies to $ku^Tu + u^TAu = k\left(\sum_{x\in V} u_x^2\right) + u^T\left(\sum_{x\sim y} u_y\right)_x = k\left(\sum_{x\in V} u_x^2\right) + \sum_{xy\in E} 2u_xu_y = \sum_{xy\in E}(u_x+u_y)^2$.
Now to prove the exercise, assume that u is an eigenvector corresponding to the eigenvalue $-k$. By the claim we get

$$0 = u^T(kI_n + A)u = \sum_{xy\in E}(u_x+u_y)^2$$

Therefore, $u_x = -u_y$ for all $xy \in E$. Assume that the graph has an odd cycle. This means we cannot have $u_x = -u_y$ for all $xy \in E$ because we will have two adjacent vertices in the odd cycle that cannot satisfy this labeling. Thus, the graph must be bipartite.

4.6.5 **Let X be a simple graph with e edges and t_3 triangles. If A is the adjacency matrix of X, show that $tr(A) = 0, tr(A^2) = 2e, tr(A^3) = 6t_3$.**

The trace of A is 0 since the number of closed walks of length one is 0 then by Theorem 4.1.2 $A(x,x) = 0$ therefore, $tr(A) = 0$.
The number of closed walks of length 2 are only the walks that start at x takes an edge to a neighboring vertex and then takes the same edge back. Therefore the number of closed walk are the edges incident to the vertex. Thus, $A(x,x) = d(x)$ and so $tr(A^2) = \sum_{x\in V} d(x) = 2e$.
The number of closed walks of length 3 are only walks that form triangles. Therefore $A(x,x)$ are the number of triangles that contain vertex x. Thus, the sum of diagonal entries will be all triangles but we count a triangle three times since it will appear three times and we can take the reverse of the triangle which doubles the amount and therefore $tr(A^3) = (2)(3)t_3 = 6t_3$.

4.6.6 **Let X be a simple graph with n vertices and e edges. If λ is an eigenvalue of the adjacency matrix A of X, show that** $|\lambda| \leq 2e(n-1)$

Let X be a simple graph with n vertices and e edges and adjacency matrix A. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A. Let λ be an arbitrary eigenvalue of A. WLOG (relabel if necessary), let $\lambda = \lambda_1$.

From 4.6.5, we know that $\mathrm{tr}(A^2) = 2e$, but the trace is also the sum of the eigenvalues. Since $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, it follows that the eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. Therefore,

$$tr(A^2) = 2e = \sum_{i=1}^{n} \lambda_i^2$$

or, equivalently,

$$2e - \lambda_1 = \sum_{i=2}^{n} \lambda_i^2 = \sum_{i=2}^{n} (-\lambda_i)^2.$$

Also, from 4.6.5, we know that $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$ which implies that $\lambda_1 = -\sum_{i=2}^{n} \lambda_i$ so

$$\lambda_1^2 = \left(-\sum_{i=2}^{n} \lambda_i\right)^2.$$

Further, by Cauchy-Schwarz,

$$\begin{aligned}\lambda_1^2 = \left(\sum_{i=2}^{n} -\lambda_i\right)^2 &\leq \left(\sum_{i=2}^{n} (-\lambda_i)^2\right)\left(\sum_{i=2}^{n} 1^2\right) \\ &= (2e - \lambda_1^2)(n-1) \\ &= 2e(n-1) - \lambda_1^2(n-1)\end{aligned}$$

and after adding $\lambda_1^2(n-1)$ to both sides,

$$\lambda_1^2 n = 2e(n-1)$$

which shows that

$$\lambda_1 = \sqrt{\frac{2e(n-1)}{n}}$$

4.6.7 **If two non-adjacent vertices of a graph X are adjacent to the same set of vertices, show that 0 is an eigenvalue of its adjacency matrix.**

Let X be a graph and let v_1 and v_2 be nonadjacent vertices that are adjacent to the same set of vertices. Let A be the adjacency matrix of X where v_1 is the first row/column and v_2 is the second row/column. Let w be the vector with 1 as the first entry, -1 as the second entry, and zero for all other entries.

Consider the j-th entry of Aw,

$$(Aw)_j = \sum_{i=1}^{n} A_{i,j} w_i = A_{1,j} - A_{2,j}.$$

since $w_i = 0$ for $2 < i \leq n$, $w_1 = 1$, and $w_2 = -1$. Lastly, since v_1 and v_2 are nonadjacent and are adjacent to the same set of vertices, $A_{1,j} = A_{2,j}$ for all $j \in [n]$. Therefore, $(Aw)_j = A_{1,j} - A_{2,j} = 0$ which means that $Aw = 0w$. In other words, w is an eigenvector of A corresponding to $\lambda = 0$. Therefore, 0 must be an eigenvalue of A.

4.6.10 **Let X = (V, E) be a graph with n vertices. If X^c is the complement of X, show that the Laplacian of X^c equals $nI_n - J_n - L(X)$. If the eigenvalues of X is $0 = \mu_0 \leq \mu_1 \leq ... \leq \mu_{n-1}$, showing that the eigenvalues of X^c are 0 and $n - \mu_j, for 1 \leq j \leq n-1$.**

The Laplacian of the graph $X = (V, E)$ equals $D(X) - A(X)$, where $D(X)$ is the diagonal degree matrix of X and $A(X)$ is the adjacency matrix of X. The Laplacian of the complement X^c of X equals $D(X^c) - A(X^c)$.

Note that $D(X) + D(X^c) = (n-1)I$ and $A(X) + A(X^c) = J - I$. Therefore,

$$\begin{aligned} L(X) + L(X^c) &= (D(X) - A(X)) + (D(X^c) - A(X^c)) \\ &= (D(X) + D(X^c)) - (A(X) + A(X^c)) \\ &= (n-1)I - (J - I) \\ &= nI - J. \end{aligned}$$

This proves the first part of the exercise.

Let $0 = \mu_0 \leq \mu_1 \leq ...\mu_{n-1}$ be the eigenvalues of $L(X)$ and let $\{\vec{u_1}, \ldots, \vec{u_n}\}$ be an orthonormal basis of $\mathbb{R}^n$ formed by the eigenvectors of $L(X)$ with

$$L(X)\vec{u_j} = \mu_j \vec{u_j}.$$

Clearly, $\vec{u_1} = \frac{\vec{1}}{\sqrt{n}}$, where $n = |V|$ is the order of X and $\vec{1}$ is the all one vector.

Also, $\langle u_1, u_j \rangle = 0$ for any $j \geq 2$, where $\langle \cdot, \cdot \rangle$ is the standard real inner product on $\mathbb{R}^n$. Therefore, if $\vec{0}$ denotes the n-dimensional all zero vector, then

$$J\vec{u_j} = \vec{0},$$

for each $j \geq 2$. For $j \geq 2$, we have that

$$\begin{aligned} L(X^c)\vec{u_j} &= (nI - J - L(X))\vec{u_j} \\ &= nI\vec{u_j} - J\vec{u_j} - L(X)\vec{u_j} \\ &= n\vec{u_j} - \vec{0} - \mu_j\vec{u_j} \\ &= (n - \mu_j)\vec{u_j}. \end{aligned}$$

Note also that

$$L(X^c)\vec{u_1} = \vec{0} = 0\vec{u_1}.$$

Because the vectors $\vec{u_1}, \ldots, \vec{u_n}$ are linearly independent, we deduce that the eigenvalues of $L(X^c)$ are

$$0 \leq n - \mu_{n-1} \leq \ldots \leq n - \mu_1.$$

2 Conclusion

These problems purposes were two-fold. To edit the *A First Course in Graph Theory and Combinatorics* textbook and to better understand eigenvalues for our Ramanujan and best graph research.